$$S = H\bar{S}H^T \tag{18}$$

From Eqs. (6) and (7) we see that

$$H = (V_2{}^I V_1{}^{-1}) \tag{19}$$

where $I$ is an identity matrix of $m$th order. Inserting this in Eq. (18) and recalling Eq. (4) gives, finally, the simple formula

$$S = VG^{-1}V^T \tag{20}$$

for the unsupported stiffness matrix of the structural element.

The matrix $G$ can be conveniently evaluated by numerical integration using a formula of the type

$$G = \sum_i \gamma_i U_i{}^T N U_i \tag{21}$$

where the $\gamma_i$ are constants and $U_i$ = the value of $U$ at some point $i$.


References

[1] Argyris, J. H., *Energy Theorems and Structural Analysis, Part 1, General Theory*, Aircraft Engineering, Eq. (84), p. 43, Feb. 1955.

[2] Timoshenko, S., and Goodier, J. N., *Theory of Elasticity*, McGraw-Hill Book Co., Inc., 2nd Ed., p. 147, Eq. (c), 1951.

[3] Turner, M. J., Clough, R. W., Martin, H. C., and Topp, L. J., *Stiffness and Deflection Analysis of Complex Structures*, Journal of the Aeronautical Sciences, Vol. 23, No. 9, Sept., 1956.

[4] Archer, J. S., and Samson, Charles, H., Jr., *Structural Idealization for Digital-Computer Analysis*, A.S.C.E., Sept. 1960, Conference Proceedings.


## On Minor-Circle Turns


R. D. Shaver*
*The RAND Corporation, Santa Monica, Calif.*
October 1, 1962


With the advent of the space age, many people have become concerned with the maneuvering characteristics of space vehicles. A great deal of literature is available on the various aspects of orbital transfers and on the various methods of providing improved performance. In a recent paper W. H. T. Loh[1] introduced a definition of a minor-circle turn. We would like to suggest at this time an alternative definition which appears useful in that it removes some of the restrictions found in the work by Loh.

Loh's definition of a minor-circle turn requires the vehicle to fly in a plane which is oriented normal to the axis of the earth and which is elevated above the equator at that distance where the plane intersects the surface of the earth at the desired latitude. While this is satisfactory for near-earth trajectories and for gentle minor-circle turns, it does not appear to be capable of handling the cases where the altitudes are large and where the latitudes of the turn approach ninety degrees. To surmount this difficulty, let us define a minor-circle trajectory as that path where the vector pointing toward the center of the earth from the vehicle always cuts the surface of the earth at a constant latitude. This is equivalent to requiring the path of the vehicle to lie on the surface of a cone whose solid angle is the supplement of twice the latitude (see Fig. 1). With this definition, we are no longer restricted to near-earth trajectories or to gentle turns but may include sharp turns with steep re-entry angles without violating our assumptions. The equations of motion may be written by inspection if we choose our three directions as normal to the surface of the cone, tangent to the velocity vector lying in the cone, and perpendicular to these two directions. Defining $\gamma$ as our ascent angle, $\alpha$ as our bank angle measured out of the cone, and $\lambda$ as our latitude, we obtain

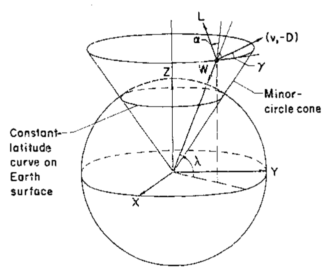


Fig. 1. Minor-circle cone.

$$(1/2)m(dv^2/ds) = m(dv/dt) = -D - mg \sin \gamma$$

$$mv^2(d\gamma/ds) = L \cos \alpha - mg \cos \gamma + (mv^2/r) \cos \gamma$$

$$(mv^2/r) \cos^2 \gamma \tan \lambda = L \sin \alpha$$

where $s$ is the distance along the path.

These three equations may be compared to those of Loh and found to be similar but not identical. For near-earth orbits of slowly varying altitudes, the equivalence is clear as is expected from the geometrical considerations of the two definitions. It is also clear that this new definition adds no new restrictions and thus the results obtained by Loh in his paper may be reproduced if desired. To demonstrate this, his Eq. (2) becomes, with our definition and notation,

$$\frac{dv^2}{ds} + \frac{2g}{(L/D)} \sqrt{\cos^2 \gamma \left\{ \left[ \frac{v^2}{gr} \cos \gamma \tan \lambda \right]^2 + \left[ 1 - \frac{v^2}{gr} + \frac{v^2}{g \cos \gamma} \left( \frac{d\gamma}{ds} \right) \right]^2 \right\}} = -2g \sin \gamma$$

and the aerodynamic control required at any moment along the minor circle becomes

$$\sin \alpha = \frac{(v^2/gr) \cos \gamma \tan \lambda}{\sqrt{\left[ \frac{v^2}{gr} \cos \gamma \tan \lambda \right]^2 + \left[ 1 - \frac{v^2}{gr} + \frac{v^2}{g \cos \gamma} \left( \frac{d\gamma}{ds} \right) \right]^2}}$$

which is similar to Eq. (3) in Loh's paper.


Reference

[1] Loh, W. H. T., *Dynamics and Thermodynamics of Re-Entry*, Journal of Aerospace Sciences, Vol. 27, No. 10, Oct. 1960.


* Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of The RAND Corp. or the official opinion or policy of any of its governmental or private research sponsors. Papers are reproduced by The RAND Corp. as a courtesy to members of its staff.

## Design of Tufts for Flow Visualization


Ascher H. Shapiro
*Ford Professor of Engineering, Massachusetts Institute of Technology, Cambridge, Mass.*
September 28, 1962


Tufts are frequently used for visualizing gas or liquid motions near solid surfaces. They are especially useful in develop-